Pseudorandom Binary Sequences from Elliptic Curves

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Definition

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Pseudorandomness

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  - well-distribution relative to arithmetic progression
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Maudit and Sárközy introduced several measures of pseudorandomness focusing on this properties.

If we want to consider a sequence as pseudorandom, then it must be indistinguishable from random sequences with respect to these measures.
Measures of pseudorandomness

Let $E_N = (e_1, \ldots, e_N) \in \{-1, +1\}^N$ be a finite binary sequence. Then

**Definition (Mauduit, Sárközy)**

The well-distribution measure of $E_N$:

$$W(E_N) = \max_{a,b,t} \left| \sum_{j=1}^{t-1} e_{a+jb} \right|,$$

where $a, b, t \in \mathbb{N}$, $a + (t - 1)b \leq N$.

The correlation measure of order $\ell$ of $E_N$:

$$C_\ell(E_N) = \max_{M,D} \left| \sum_{j=1}^{M} e_{n+d_1} e_{n+d_2} \ldots e_{n+d_\ell} \right|,$$

$D = (d_1, d_2, \ldots, d_\ell)$, $M \in \mathbb{N}$, $M + d_\ell \leq N$. 
Measures of pseudorandomness

Theorem (Alon, Kohayakava, Mauduit, Moreira, Rödl)

If $E_N$ is a truly random sequence, then we have

$$\frac{1}{\delta} \sqrt{N} < W(E_N) < \delta \sqrt{N}$$

and

$$\frac{2}{5} \sqrt{N \log \left( \frac{N}{\ell} \right)} < C_\ell(E_N) < \frac{7}{4} \sqrt{N \log \left( \frac{N}{\ell} \right)}.$$

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The $E_N$ sequence is considered as a pseudorandom sequence if

$$W(E_N) \ll N^{1/2} \log N^c \quad \text{ill.} \quad C_{\ell}(E_N) \ll \ell N^{1/2} \log N^c'.$$
Earlier construction over $\mathbb{F}_p$

Several construction have been tested in terms of these measures earlier:

- **Goubin, Mauduit, Sárközy**: Legendre symbol sequence:
  \[
e_n = \left( \frac{f(n)}{p} \right)
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- **Gyarmati**: Construction based on the discrete logarithm:
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- **Gyarmati, Pethő, Sárközy**: A transform of sequences generating by linear recursion:
  \[ x_n \equiv c_1 x_{n-1} + c_2 x_{n-2} + \cdots + c_h x_{n-h} \pmod{p} \]
  \[ e_n = \left( \frac{x_n}{p} \right) \]

Here \( f \in \mathbb{F}_p[x] \), \( \left( \frac{\cdot}{p} \right) \) is the Legendre symbol modulo \( p \).
Elliptic curves

\[ E(\mathbb{F}_p) = \{ (x, y) : y^2 = x^3 + Ax + B \}, \quad A, B \in \mathbb{F}_p. \]
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- The number of points in \(E(\mathbb{F}_p)\) satisfies:

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- The number of points in \(\mathcal{E}(\mathbb{F}_p)\) satisfies:
  \[ |p + 1 - |\mathcal{E}(\mathbb{F}_p)|| \leq 2\sqrt{q}. \]
- The set of rational function \(f(P)(= f(x, y))\) on \(\mathcal{E}\) is
  \[ \mathbb{F}_p(\mathcal{E}) = \mathbb{F}_p(x, y)/(y^2 = x^3 + Ax + B). \]
- We will use the notation: \(P = (x(P), y(P))\).
Sequences generated from Elliptic Curves

Let $G$ be a generator of $E(\mathbb{F}_p)$ (or at least an element with large order). Then

$$n \mapsto x(nG) \in \mathbb{F}_p,$$

where $\mathbb{F}_p$ is the Legendre symbol modulo $p$. 

Chen: $n \mapsto \left( \frac{f(nG)}{p} \right)$

Liu, Zhan, Wang: $n \mapsto \begin{cases} +1 & \text{if } f(nG) \in \{0, 1, 2, \ldots, p-1\} \\ -1 & \text{otherwise} \end{cases}$
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Sequences generated from Elliptic Curves, example

Let \( \mathcal{E} : y^2 = x^3 - 2x \) over \( \mathbb{F}_{19} \)
\(|\mathcal{E}| = 20\), \( G = (2, 2) \) is a generator.
Let \( f(x, y) = x \).
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Let \( nG = (x, y) \), then

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e_n \cdot e_{n+10} = \left( \frac{f(nG)}{19} \right) \cdot \left( \frac{f((n+10)G)}{19} \right) = \left( \frac{f(nG) \cdot f(nG + 10G)}{19} \right)
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= \left( \frac{f(nG) \cdot f(nG + (0, 0))}{19} \right) = \left( \frac{x \cdot \left( \frac{y}{x} \right)^2 - x}{19} \right) = 1,
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since \( x \cdot \left( \frac{y}{x} \right)^2 - x \equiv -2 \mod y^2 = x^3 - 2x \) over \( \mathbb{F}_{19} \) is a constant function!
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is a constant function!
Thus the original sequence

$$n \mapsto x(nG) \in \mathbb{F}_p,$$
is also not pseudorandom.
Sequences generated from Elliptic Curves, admissibility

In general: the $C_{\ell}(E_T)$ is small, if the function

$$F(P) = f(P + d_1 G) \ldots f(P + d_\ell G) \in \mathbb{F}_p(E)$$

is not a square.
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Definition

$(k, \ell, m)$ is a $d$-admissible triple, if there are no multisets $\mathcal{A}, \mathcal{B} \subset \mathbb{Z}_m$ such that

- $|\mathcal{A}| = k, |\mathcal{B}| = \ell$
- the number of solution of $a + b = c$, $a \in \mathcal{A}, b \in \mathcal{B}$ is divisible by $d$. 

Let $k = |\text{Supp}(f)|, m = p, d = 2$. If the triple $(\text{Supp}(f), \ell, p)$ is 2-admissible, then the function $F$ is not a square.

A: the multiset of the zeros and poles of $f$;

B = $\{d_1 G, \ldots, d_\ell G\}$;

$F$ is a square, then $a + b = c$ has even number of solution for each $c$. 
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- $\mathcal{A}$: the multiset of the zeros and poles of $f$;
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Sequences generated from Elliptic Curves, general construction

**Construction (Chen)**

Let $G$ be a generator of $E(\mathbb{F}_p)$, and $f \in \mathbb{F}_p(E)$, and let us define $E_T = (e_1, \ldots, e_T)$ by

$$e_n = \begin{cases} \left( \frac{f(nG)}{p} \right) + 1 & \text{ha } f(nG) \neq 0, \mathcal{O}, \\ 0 & \text{ha } f(nG) = 0, \mathcal{O}. \end{cases}$$
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Construction (Mérai)

Let $G$ be a generator of $\mathcal{E}(\mathbb{F}_p)$, and $f \in \mathbb{F}_p(\mathcal{E})$, $\chi$ is a multiplicative character of $\mathbb{F}_p$, and let us define $E_T = (e_1, \ldots, e_T)$ by

$$e_n = \begin{cases} +1 & \text{if } \arg(\chi(f(nG))) \in [0, \pi) \\ -1 & \text{otherwise.} \end{cases}$$
Sequences generated from Elliptic Curves, general construction

**Theorem**

If the order of $G$ is $T$, and the order of $\chi$ is $d$ then

$$W(E_T) \ll |\text{Supp}(f)|p^{1/2}(1 + \log T)\log d.$$  

If the triple $(|\text{Supp}(f)|, \ell, T)$ is $d$-admissible, then

$$C_\ell(E_T) \ll \ell 10^\ell |\text{Supp}(f)|p^{1/2}(1 + \log T)(\log d)^\ell.$$
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**Special cases:**

- $d = 2$: Legendre symbol sequence over elliptic curves.
- $d = p - 1$: Chen, Xiao: Elliptic curve analogue of a construction of Gyarmati based on the discrete logarithm.
Admissibility

Theorem

Let $p(m)$ be the smallest prime factor of $m$. Then

- If $k < p(m)$, then the triple $(k, 2, m)$ is $d$-admissible.
- If $(4\ell)^k < p(m)$,
  then $(k, \ell, m)$ is $d$-admissible.
- If $m$ is a prime, and each prime factor of $d$ is primitive root modulo $m$, then $(k, \ell, m)$ is $d$-admissible.

Note: It is enough to prove the theorem in the case when $d$ is a prime number.
Proof of the admissibility I.

If there exist multisets $\mathcal{A}, \mathcal{B}$, such that

- $|\mathcal{A}| = k$, $|\mathcal{B}| = 2$;
- for each $c$ if the equation $a + b = c$ has solution, then there are at least two.

Let $\mathcal{B} = \{r, r + s\}$ ($s \neq 0$).

Then each element of $\mathcal{A} + r$ has at least two representations.

So

$$|\mathcal{A}| = |\{a + r \mid a \in \mathcal{A}\}| = |\{a + r + s \mid a \in \mathcal{A}\}| =$$

$$= |\{a + r + st \mid a \in \mathcal{A}\}| \geq p(m),$$

since $\{a + r + st \mid a \in \mathcal{A}\}$ is a non-trivial co-set of $\mathbb{Z}_m$, which contradicts to the condition $k < p(m)$. 

Proof of the admissibility III.

Let $p = m$, $d$ be prime numbers. For a given multiset $C \subseteq \mathbb{Z}_p$ let

$$P_C(x) = \sum_{c \in C} x^{r_p(c)}.$$  

(where $r_m(c)$ is the least non-negative residue of $c$ modulo $p$.)

For a given $u \in \mathbb{Z}_p$ we have

$$P_{u+C}(x) \equiv x^u \cdot P_C(x) \mod x^p - 1 \text{ over } \mathbb{Z}_d.$$  

In $A + B$ each element is represented in $d$ ways if and only if

$$P_A(x) \cdot P_B(x) \equiv P_{A+B}(x) = 0 \mod x^p - 1 \text{ over } \mathbb{Z}_d.$$  

So there are no multisets $A, B$ if the polynomial

$$\frac{x^p - 1}{x - 1} = x^{p-1} + \cdots + 1$$  

is irreducible over $\mathbb{Z}_d$, i.e. $d$ is primitive root modulo $p$. 
Sequences generated from Elliptic Curves

Let $G$ be a generator of $E(\mathbb{F}_p)$ (or at least an element with large order). Then

$$n \mapsto x(nG) \in \mathbb{F}_p,$$

or in general

$$n \mapsto f(nG) \in \mathbb{F}_p,$$

where $f \in \mathbb{F}_p(E)$.

In order to generate binary sequences we have to choose one of the bits of $f(nG)$:

Chen:

$$n \mapsto \left( \frac{f(nG)}{p} \right)$$

Liu, Zhan, Wang:

$$n \mapsto \begin{cases} +1 & \text{if } f(nG) \in \{0, 1, 2 \ldots, \frac{p-1}{2} \} \\ -1 & \text{otherwise} \end{cases}$$

Here $\left( \frac{\cdot}{p} \right)$ is the Legendre symbol modulo $p$. 
An other construction

Construction

Let $G$ be a generator of $\mathcal{E}(\mathbb{F}_p)$, and $f \in \mathbb{F}_p(\mathcal{E})$, and let us define $E_T = (e_1, \ldots, e_T)$ by

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Liu, Zhan, Wang:

- $f$ is a "polynomial", i.e. the $0$ is the only pole of $f$;
- $1/f$ is a "polynomial", i.e. the $0$ is the only zero of $f$;

What can we say, when $f$ is a general function?
An other construction, an example

Again let \( \mathcal{E} : y^2 = x^3 - 2x \) over \( \mathbb{F}_{19} \)
\(|\mathcal{E}| = 20\), \( G = (2, 2) \) is a generator.

Let \( f(x, y) = 9x + \frac{1}{x} \).

\[
e_n = \begin{cases} 
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<td>-1</td>
<td>12</td>
<td>(16,6)</td>
<td>17</td>
<td>-1</td>
</tr>
<tr>
<td>3</td>
<td>(15,1)</td>
<td>16</td>
<td>-1</td>
<td>13</td>
<td>(10,12)</td>
<td>16</td>
<td>-1</td>
</tr>
<tr>
<td>4</td>
<td>(11,6)</td>
<td>11</td>
<td>-1</td>
<td>14</td>
<td>(5,18)</td>
<td>11</td>
<td>-1</td>
</tr>
<tr>
<td>5</td>
<td>(13,10)</td>
<td>6</td>
<td>+1</td>
<td>15</td>
<td>(13,9)</td>
<td>6</td>
<td>+1</td>
</tr>
<tr>
<td>6</td>
<td>(5,1)</td>
<td>11</td>
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<td>16</td>
<td>(11,13)</td>
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<tr>
<td>7</td>
<td>(10,7)</td>
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<td>17</td>
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</tr>
<tr>
<td>8</td>
<td>(16,13)</td>
<td>17</td>
<td>-1</td>
<td>18</td>
<td>(7,5)</td>
<td>17</td>
<td>-1</td>
</tr>
<tr>
<td>9</td>
<td>(18,18)</td>
<td>9</td>
<td>+1</td>
<td>19</td>
<td>(2,17)</td>
<td>9</td>
<td>+1</td>
</tr>
<tr>
<td>10</td>
<td>(0,0)</td>
<td>( \infty )</td>
<td>-1</td>
<td>20</td>
<td>( \emptyset )</td>
<td>( \infty )</td>
<td>-1</td>
</tr>
</tbody>
</table>
An other construction, an example

Again let \( \mathcal{E} : y^2 = x^3 - 2x \) over \( \mathbb{F}_{19} \)

\(|\mathcal{E}| = 20, \ G = (2, 2) \) is a generator.

Let \( f(x, y) = 9x + \frac{1}{x} \).

\[
e_n = \begin{cases} 
+1 & 9x(nG) + \frac{1}{x(nG)} \in \{0, 1, 2, \ldots, 9\} \\
-1 & \text{otherwise.}
\end{cases}
\]

Why this sequence is \emph{not} random?
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Why this sequence is not random? The correlation measure $C_2(E_{20})$ is large:

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e_n \cdot e_{n+10} = +1 \quad \text{for } n = 1, 2, \ldots, 10.$$
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By using additive character sums, it can be shown that the correlation measure of order $\ell$ is small, if none of the functions

$$F(P) = h_1 \cdot f(P + d_1 G) + \cdots + h_\ell \cdot f(P + d_\ell G) \quad (h_1, \ldots, h_\ell) \in \mathbb{F}_p^\ell \setminus (0, \ldots, 0)$$

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But

$$f(P) - f(P + 10G) = \left(9x + \frac{1}{x}\right) - \left(9 \left(\frac{y}{x}\right)^2 - x\right) + \frac{1}{\left(\frac{y}{x}\right)^2 - x}$$

$$\begin{align*}
&= \left(9x + \frac{1}{x}\right) - \left(9 \cdot \frac{-2}{x} + \frac{-1}{2} \cdot x\right) = 0
\end{align*}$$
**Construction (Mérai)**

Let $G$ be a generator of $\mathcal{E}(\mathbb{F}_p)$, and $f \in \mathbb{F}_p(\mathcal{E})$, and let us define $E_T = (e_1, \ldots, e_T)$ by

$$e_n = \begin{cases} 
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For general rational functions:

Theorem

If the order of $G$ is $T$ then

$$W(E_T) \ll |\operatorname{Supp}(f)| p^{1/2} \log p \log T.$$  

If $p(T)$ is the least prime divisor of $p$ and

- $|\operatorname{Supp}(f)| < p(T)$ and $\ell = 2$, or
- $(4|\operatorname{Supp}(f)|)^\ell < p(T)$,

then

$$C_\ell(E_T) \ll \ell |\operatorname{Supp}(f)| p^{1/2} (\log p)^{\ell+1} \log T.$$
Extension of constructions to several dimensions

The extension of binary sequences in *several dimensions*, called *binary lattice*:

\[ \eta : \{1, 2, \ldots, N\}^n \rightarrow \{+1, -1\} \]

We can define the analogue of the measures of pseudorandomness in several dimensions: \( Q_\ell(\eta) \).

Application:
- encryption of several dimensions map or picture via the analogue of the Vernam cipher.
Let $q = p^n$ be a prime power, $u_1, \ldots, u_n \in \mathbb{F}_q$ is a basis over $\mathbb{F}_p$. Let $f \in \mathbb{F}_q[x]$. 

Huber, Mauduit, Sárközy:

$\eta(x_1, \ldots, x_n) = \chi_2(f(x_1u_1 + \cdots + x_nu_n))$, where $\chi_2$ is the quadratic character over $\mathbb{F}_q$.

Mérai:

$\eta(x_1, \ldots, x_n) = +1$, if $\arg(\chi(f(x_1u_1 + \cdots + x_nu_n))) \in [0, \pi)$, where $\chi$ is a general multiplicative character.

Mauduit, Sárközy:

$\eta(x_1, \ldots, x_n) = f^{-1}(x_1u_1 + \cdots + x_nu_n) \in B$, where $x^{-1}$ is the multiplicative inverse of $x$, $B \subset \mathbb{F}_q$. 
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Constructions of binary lattices

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Construction of binary lattice from elliptic curves

Definition

The points $P_1, \ldots, P_n \in E$ are weakly independents if

$$\lambda_1 P_1 + \cdots + \lambda_n P_n = \mathcal{O} \Rightarrow \lambda_i P_i = \mathcal{O} \text{ for each } i = 1, \ldots, n.$$
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Construction (Mérai)

Let $P_1, \ldots, P_n \in \mathcal{E}$ are weakly independent element, let us define $\eta$ by

$$\eta(x_1, \ldots, x_n) = \begin{cases} 
\left( \frac{f(x_1 P_1 + \cdots + x_n P_n)}{p} \right) & \text{if } x_1 P_1 + \cdots + x_n P_n \neq \mathcal{O} \\
-1 & \text{otherwise}.
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Example

- If \( \mathcal{E} \) is not cyclic, and \( P, Q \in \mathcal{E} \) are the echelonized generators, then they are weakly independents.
Construction of binary lattice from elliptic curves

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**Example**

- If \( \mathcal{E} \) is not cyclic, and \( P, Q \in \mathcal{E} \) are the echelonized generators, then they are weakly independents.
- If \( P \in \mathcal{E}, |P| = \alpha_1 \ldots \alpha_n \) such that the numbers \( \alpha_1 \ldots \alpha_n \) are pairwise co-prime, then the elements

\[
P_1 = \frac{|P|}{\alpha_1} P, \ldots, P_n = \frac{|P|}{\alpha_n} P
\]

are weakly independents.
Construction of binary lattice from elliptic curves

\[ \eta(x_1, \ldots, x_n) = \begin{cases} \left( \frac{f(x_1 P_1 + \cdots + x_n P_n)}{p} \right) & \text{if } x_1 P_1 + \cdots + x_n P_n \neq \mathcal{O} \\ -1 & \text{otherwise.} \end{cases} \]

**Theorem**

Let \( \mathcal{H} \) be the subgroup generated by \( P_1, \ldots, P_n \), \( p(\mathcal{H}) \) is the least prime divisor of \( |\mathcal{H}| \). If

- \( |\text{Supp}(f)| < p(\mathcal{H}) \) and \( \ell = 2 \); or
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\[ Q_\ell \ll_{n, \ell, f} p^{1/2 + \varepsilon}. \]
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- Good constructions can be defined with general multiplicative characters.
- The proof based on the notion of admissibility over general (not cyclic) Abelian group and character sum estimates over elliptic curves.
TOUR OF ACCOUNTING

OVER HERE. WE HAVE OUR RANDOM NUMBER GENERATOR.

NINE NINE NINE NINE NINE

ARE YOU SURE THAT'S RANDOM?

THAT'S THE PROBLEM WITH RANDOMNESS: YOU CAN NEVER BE SURE.
Thank You!

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