Pseudorandom Binary Sequences from Elliptic Curves

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Maudit and Sárközy introduced several measures of pseudorandomness focusing on this properties.

If we want to consider a sequence as pseudorandom, then it must be indistinguishable from random sequences with respect to these measures.

Measures of pseudorandomness

Let $E_N = (e_1, \dots, e_N) \in \{-1, +1\}^N$ be a finite binary sequence. Then

Definition (Mauduit, Sárközy)

The well-distribution measure of E_N:

$$W(E_N) = \max_{a,b,t} \left| \sum_{j=1}^{t-1} e_{a+jb} \right|,$$

where $a, b, t \in \mathbb{N}$, $a + (t-1)b \leq N$.

The correlation measure of order ℓ of E_N:

$$C_\ell(E_N) = \max_{M,D} \left| \sum_{j=1}^{M} e_{n+d_1} e_{n+d_2} \dots e_{n+d_\ell} \right|,$$

$D = (d_1, d_2, \dots, d_\ell)$, $M \in \mathbb{N}$, $M + d_\ell \leq N$.

Measures of pseudorandomness

Theorem (Alon, Kohayakava, Mauduit, Moreira, Rödl)

If E_N is a truly random sequence, then we have

$$\frac{1}{\delta}\sqrt{N} < W(E_N) < \delta\sqrt{N}$$

and

$$\frac{2}{5}\sqrt{N\log\binom{N}{\ell}} < C_\ell(E_N) < \frac{7}{4}\sqrt{N\log\binom{N}{\ell}}.$$

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Definition

The E_N sequence is considered as a pseudorandom sequence if

$$W(E_N) \ll N^{1/2}\log N^{c} \quad \text{ill.} \quad C_\ell(E_N) \ll \ell N^{1/2}\log N^{c'}.$$

Earlier construction over $\mathbb{F}_p$

Several construction have been tested in terms of these measures earlier:

- *Goubin, Mauduit, Sárközy*: Legendre symbol sequence:

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- *Gyarmati, Pethő, Sárközy*: A transform of sequences generating by linear recursion:

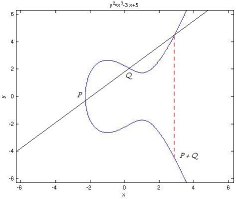
$$x_n \equiv c_1 x_{n-1} + c_2 x_{n-2} + \cdots + c_h x_{n-h} \pmod p$$

$$e_n = \left(\frac{x_n}{p}\right)$$

Here $f \in \mathbb{F}_p[x]$, $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol modulo p.

Elliptic curves

$$\mathcal{E}(\mathbb{F}_p) = \{(x, y) : y^2 = x^3 + Ax + B\}, \quad A, B \in \mathbb{F}_p.$$



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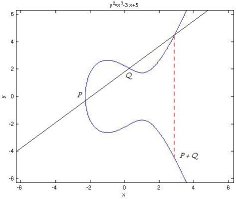

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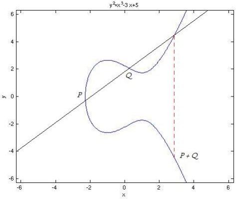

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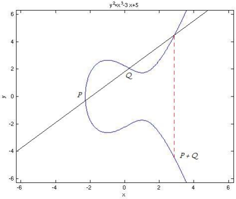

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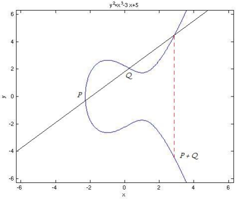

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- The number of points in $\mathcal{E}(\mathbb{F}_p)$ satisfies:

$$|p + 1 - |\mathcal{E}(\mathbb{F}_p)|| \leq 2\sqrt{q}.$$

- The set of rational function $f(P)(= f(x, y))$ on $\mathcal{E}$ is

$$\mathbb{F}_p(\mathcal{E}) = \mathbb{F}_p(x, y)/(y^2 = x^3 + Ax + B).$$

- We will use the notation: $P = (x(P), y(P))$.

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Sequences generated from Elliptic Curves, example

Let $\mathcal{E} : y^2 = x^3 - 2x$ over $\mathbb{F}_{19}$

$|\mathcal{E}| = 20$, $G = (2, 2)$ is a generator.

Let $f(x, y) = x$.

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5	(13,10)		15	(13,9)	
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is a constant function!
Thus the original sequence

$$n \longmapsto x(nG) \in \mathbb{F}_p,$$

is also not pseudorandom.

Sequences generated from Elliptic Curves, admissibility

In general: the $C_\ell(E_T)$ is small, if the function

$$F(P) = f(P + d_1 G) \dots f(P + d_\ell G) \in \mathbb{F}_p(\mathcal{E})$$

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Definition

(k, ℓ, m) is a d-admissible triple, if there are no multisets $\mathcal{A}, \mathcal{B} \subset \mathbb{Z}_m$ such that

- $|\mathcal{A}| = k$, $|\mathcal{B}| = \ell$
- *the number of solution of $a + b = c$, $a \in \mathcal{A}$, $b \in \mathcal{B}$ is divisible by d.*

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Let $k = |\operatorname{Supp}(f)|$, $m = p$, $d = 2$. If the triple $(\operatorname{Supp}(f), \ell, p)$ is 2-admissible, then the function F is *not* a square.
If

- $\mathcal{A}$: the multiset of the zeros and poles of f;
- $\mathcal{B} = \{d_1 G, \dots, d_\ell G\}$;
- F is a square,

then $a + b = c$ has even number of solution for each c.

Sequences generated from Elliptic Curves, general construction

Construction (Chen)

Let G be a generator of $\mathcal{E}(\mathbb{F}_p)$*, and* $f \in \mathbb{F}_p(\mathcal{E})$*, and let us define* $E_T = (e_1, \ldots, e_T)$ *by*

$$e_n = \begin{cases} \left(\frac{f(nG)}{p}\right) & \text{ha } f(nG) \neq 0, \mathcal{O}, \\ +1 & \text{ha } f(nG) = 0, \mathcal{O}. \end{cases}$$

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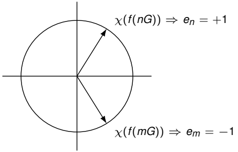
Construction (Mérai)

Let G be a generator of $\mathcal{E}(\mathbb{F}_p)$, and $f \in \mathbb{F}_p(\mathcal{E})$, χ is a multiplicative character of $\mathbb{F}_p$, and let us define $E_T = (e_1, \ldots, e_T)$ by

$$e_n = \begin{cases} +1 & \text{if } \arg\left(\chi(f(nG))\right) \in [0, \pi) \\ -1 & \text{otherwise.} \end{cases}$$



Sequences generated from Elliptic Curves, general construction

Theorem

If the order of G is T, and the order of χ *is d then*

$$W(E_T) \ll |\operatorname{Supp}(f)|p^{1/2}(1+\log T)\log d.$$

If the triple $(|\operatorname{Supp}(f)|, \ell, T)$ *is d-admissible, then*

$$C_\ell(E_T) \ll \ell 10^\ell |\operatorname{Supp}(f)|p^{1/2}(1+\log T)(\log d)^\ell.$$

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- $d = p - 1$: Chen, Xiao: Elliptic curve analogue of a construction of Gyarmati based on the discrete logarithm.

Admissibility

Theorem

Let $p(m)$ be the smallest prime factor of m. Then

- *If $k < p(m)$, then the triple $(k, 2, m)$ is d-admissible.*
- *If*

  $$(4\ell)^k < p(m),$$

  then (k, ℓ, m) is d-admissible.
- *If m is a prime, and each prime factor of d is primitive root modulo m, then (k, ℓ, m) is d-admissible.*

Note: It is enough to prove the theorem in the case when d is a prime number.

Proof of the admissibility I.

If there exist multisets $\mathcal{A}, \mathcal{B}$, such that

- $|\mathcal{A}| = k$, $|\mathcal{B}| = 2$;
- for each c if the equation $a + b = c$ has solution, then there are at least two.

Let $\mathcal{B} = \{r, r + s\}$ ($s \neq 0$).
Then each elements of $\mathcal{A} + r$ has at least two representations

So

$$|\mathcal{A}| = |\{a + r \mid a \in \mathcal{A}\}| = |\{a + r + s \mid a \in \mathcal{A}\}| = \\ = |\{a + r + st \mid a \in \mathcal{A}\}| \geq p(m),$$

since $\{a + r + st \mid a \in \mathcal{A}\}$ is a not-trivial co-set of $\mathbb{Z}_m$, which contradicts to the condition $k < p(m)$.

Proof of the admissibility III.

Let $p = m$, d be prime numbers.
For a given multiset $\mathcal{C} \subseteq \mathbb{Z}_p$ let

$$P_{\mathcal{C}}(x) = \sum_{c \in \mathcal{C}} x^{r_p(c)}.$$

(where $r_m(c)$ is the least non-negative residue of c modulo p.)

For a given $u \in \mathbb{Z}_p$ we have

$$P_{u+\mathcal{C}}(x) \equiv x^u \cdot P_{\mathcal{C}}(x) \mod x^p - 1 \text{ over } \mathbb{Z}_d.$$

In $\mathcal{A} + \mathcal{B}$ each element is represented in d ways if and only if

$$P_{\mathcal{A}}(x) \cdot P_{\mathcal{B}}(x) \equiv P_{\mathcal{A}+\mathcal{B}}(x) = 0 \mod x^p - 1 \text{ over } \mathbb{Z}_d.$$

So there are *no* multisets $\mathcal{A}, \mathcal{B}$ if the polinomial

$$\frac{x^p - 1}{x - 1} = x^{p-1} + \cdots + 1$$

is irreducible over $\mathbb{Z}_d$, i.e. d is primitive root modulo p.

Sequences generated from Elliptic Curves

Let G be a generator of $\mathcal{E}(\mathbb{F}_p)$ (or at least an element with large order). Then

$$n \longmapsto x(nG) \in \mathbb{F}_p,$$

or in general

$$n \longmapsto f(nG) \in \mathbb{F}_p,$$

where $f \in \mathbb{F}_p(\mathcal{E})$.
In order to generate *binary* sequences we have to choose one of the bits of $f(nG)$:

Chen: $\quad n \mapsto \left(\frac{f(nG)}{p}\right)$

Liu, Zhan, Wang: $\quad n \mapsto \begin{cases} +1 & \text{if } f(nG) \in \{0, 1, 2 \ldots, \frac{p-1}{2}\} \\ -1 & \text{otherwise} \end{cases}$

Here $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol modulo p.

An other construction

Construction

Let G be a generator of $\mathcal{E}(\mathbb{F}_p)$, and $f \in \mathbb{F}_p(\mathcal{E})$, and let us define $E_T = (e_1, \ldots, e_T)$ by

$$e_n = \begin{cases} +1 & f(nG) \in \{0, 1, 2, \ldots, \frac{p-1}{2}\} \\ -1 & \text{otherwise.} \end{cases}$$

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Liu, Zhan, Wang:

- f is a "polynomial", i.e. the $\mathcal{O}$ is the only pole of f;
- $1/f$ is a "polynomial", i.e. the $\mathcal{O}$ is the only zero of f;

What can we say, when f is a general function?

An other construction, an example

Again let $\mathcal{E} : y^2 = x^3 - 2x$ over $\mathbb{F}_{19}$

$|\mathcal{E}| = 20$, $G = (2, 2)$ is a generator.

Let $f(x, y) = 9x + \frac{1}{x}$.

$$e_n = \begin{cases} +1 & 9x(nG) + \frac{1}{x(nG)} \in \{0, 1, 2, \ldots, 9\} \\ -1 & \text{otherwise.} \end{cases}$$

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n	nG	$f(nG)$	e_n
1	(2,2)	9	
2	(7,14)	17	
3	(15,1)	16	
4	(11,6)	11	
5	(13,10)	6	
6	(5 ,1)	11	
7	(10 ,7)	16	
8	(16 ,13)	17	
9	(18 ,18)	9	
10	(0 ,0)	∞	

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15	(13,9)	6	
16	(11,13)	11	
17	(15,18)	16	
18	(7,5)	17	
19	(2,17)	9	
20	$\mathcal{O}$	∞	

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$$F(P) = h_1 \cdot f(P + d_1 G) + \cdots + h_\ell \cdot f(P + d_\ell G) \quad (h_1, \dots, h_\ell) \in \mathbb{F}_p^\ell \setminus (0, \dots, 0)$$

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are constant.
But

$$f(P) - f(P+10G) = \left(9x + \frac{1}{x}\right) - \left(9\left(\left(\frac{y}{x}\right)^2 - x\right) + \frac{1}{\left(\frac{y}{x}\right)^2 - x}\right)$$
$$= \left(9x + \frac{1}{x}\right) - \left(9 \cdot \frac{-2}{x} + \frac{-1}{2}x\right) = 0$$

An other construction

Construction (Mérai)

Let G be a generator of $\mathcal{E}(\mathbb{F}_p)$, and $f \in \mathbb{F}_p(\mathcal{E})$, and let us define $E_T = (e_1, \ldots, e_T)$ by

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For general rational functions:

Theorem

If the order of G is T then

$$W(E_T) \ll |\operatorname{Supp}(f)| p^{1/2} \log p \log T.$$

If $p(T)$ is the least prime divisor of p and

- $|\operatorname{Supp}(f)| < p(T)$ *and* $\ell = 2$, *or*
- $(4|\operatorname{Supp}(f)|)^{\ell} < p(T)$,

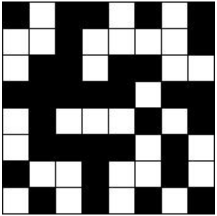
then

$$C_\ell(E_T) \ll \ell |\operatorname{Supp}(f)| p^{1/2} (\log p)^{\ell+1} \log T.$$

Extension of constructions to several dimensions

The extension of binary sequences in *several dimensions*, called *binary lattice*:

$$\eta : \{1, 2, \ldots, N\}^n \to \{+1, -1\}$$



We can define the analogue of the measures of pseudorandomness in several dimensions: $Q_\ell(\eta)$.

Application:

- encryption of several dimensions map or picture via the analogue of the Vernam cipher.

Constructions of binary lattices

Let $q = p^n$ be a prime power, $u_1, \ldots, u_n \in \mathbb{F}_q$ is a basis over $\mathbb{F}_p$. Let $f \in \mathbb{F}_q[x]$

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$$\eta(x_1, \dots, x_n) = \chi_2\left(f(x_1 u_1 + \dots + x_n u_n)\right),$$

where χ_2 is the quadratic character over $\mathbb{F}_q$.

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$$\eta(x_1, \dots, x_n) = +1, \text{ if } \arg\Big(\chi\big(f(x_1 u_1 + \dots + x_n u_n)\big)\Big) \in [0, \pi),$$

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- Mauduit, Sárközy:

$$\eta(x_1, \dots, x_n) = f^{-1}(x_1 u_1 + \dots + x_n u_n) \in B,$$

  where x^{-1} is the multiplicative inverse of x, $B \subset \mathbb{F}_q$.

Construction of binary lattice from elliptic curves

Definition

The points $P_1, \ldots, P_n \in \mathcal{E}$ are weakly independents if

$$\lambda_1 P_1 + \cdots + \lambda_n P_n = \mathcal{O} \Rightarrow \lambda_i P_i = \mathcal{O} \text{ for each } i = 1, \ldots, n.$$

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$$\eta(x_1, \ldots, x_n) = \begin{cases} \left(\frac{f(x_1 P_1 + \cdots + x_n P_n)}{p}\right) & \text{if } x_1 P_1 + \cdots + x_n P_n \neq \mathcal{O} \\ -1 & \text{otherwise.} \end{cases}$$

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- If $\mathcal{E}$ is *not* cyclic, and $P, Q \in \mathcal{E}$ are the echelonized generators, then they are weakly independents.

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Example

- If $\mathcal{E}$ is *not* cyclic, and $P, Q \in \mathcal{E}$ are the echelonized generators, then they are weakly independents.
- If $P \in \mathcal{E}$, $|P| = \alpha_1 \ldots \alpha_n$ such that the numbers $\alpha_1 \ldots \alpha_n$ are pairwise co-prime, then the elements

  $$P_1 = \frac{|P|}{\alpha_1} P, \ldots, P_n = \frac{|P|}{\alpha_n} P$$

  are weakly independents.

Construction of binary lattice from elliptic curves

$$\eta(x_1, \dots, x_n) = \begin{cases} \left(\frac{f(x_1P_1+\cdots+x_nP_n)}{p}\right) & \text{if } x_1P_1 + \cdots + x_nP_n \neq \mathcal{O} \\ -1 & \text{otherwise.} \end{cases}$$

Theorem

Let $\mathcal{H}$ be the subgroup generated by $P_1, \dots, P_n$, $p(\mathcal{H})$ is the least prime divisor of $|\mathcal{H}|$. If

- $|\operatorname{Supp}(f)| < P(\mathcal{H})$ *and* $\ell = 2$*; or*
- $4^{n(|\operatorname{Supp}(f)|+\ell)} < p(\mathcal{H})$,

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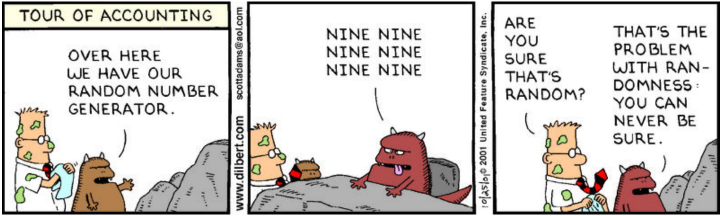
- Good constructions can be defined with general multiplicative characters.
- The proof based on the notion of admissibility over general (not cyclic) Abelian group and character sum estimates over elliptic curves.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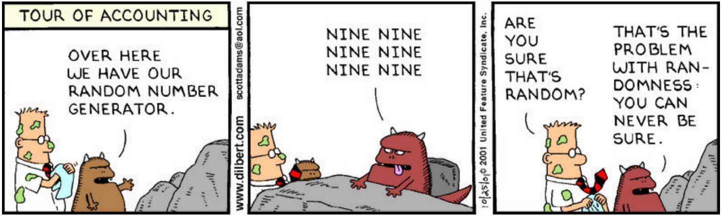

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