

Group Theory and Cryptography

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Joint work with Carlos Cid, Ciaran Mullan



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Overview

- 1 The Discrete Log Problem
- 2 The DLP and groups
- 3 Logarithmic signatures
- 4 MST_3

Discrete logarithm problem

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- When $G = \mathbb{Z}_p^*$, this seems to be a hard computational problem (for large p).
- Taking G the group of points of an elliptic curve is thought to be more secure (no 'index-calculus').
- The DLP makes sense for any group. But it's really about cyclic groups, since we can replace G by $\langle g \rangle$.

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- But not always hard: $G = (\mathbb{Z}_p, +)$.
- Another bad idea: $G \leq \text{GL}(n, q)$.
- But what about non-abelian groups?

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- Ko et al. suggest using a braid group.

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- Length based attacks work for many instances.
- How can we generate hard instances?
- There are no braid based schemes that are competitive with (say) elliptic curve DLP-based schemes.

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Covers

- A **cover** is an (ordered) collection of sets

$$A_1, A_2, \dots, A_s \subseteq G$$

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- This generalises the DLP:

$$A_i = \{1, g^{2^i}\} \text{ for } 1 \leq i \leq \log_2 |\langle g \rangle|.$$

is a cover for $\langle g \rangle$.

Logarithmic signatures

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- More general example (**transversal** logarithmic signatures) : Given a chain

$$1 = H_0 < H_1 < \cdots < H_s = G$$

of subgroups, set A_i to be a complete set of coset representatives for H_i in H_{i-1} .

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- If we have a cover, the function $\check{\alpha}$ makes sense and is onto, but is not necessarily injective.

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- To **Decrypt** (y_1, y_2) : use $y_2 = \check{\beta}(x) t^{-1}\check{\alpha}(x)t$.

Platform groups and practicalities

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- **Unanswered question:** How to generate B_1, B_2, \dots, B_5 .

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- There are $|G| = q^2$ possibilities for t . But we only need to check $|G/Z| = q$ possibilities: it's only t modulo Z that matters.

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A simple attack

- Lempken et al. gave several $O(q^2)$ attacks. Magliveras, Svaba, van Trung and Zajac give the following attack.
- We are given covers A_i and G_i , where

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- This is an exponential attack. It works for any group G .
- More practical attacks use knowledge of how the B_i are generated.

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- Do some of the following operations:
 - ▶ **Shift**: replace X_i by $X_i z$ for some $z \in Z$.
 - ▶ **Permute**: swap X_i and X_j ; permute the elements in X_i .
 - ▶ **Amalgamate**: replace X_i and X_j by

$$X_i X_j = \{x_i x_j \mid x_i \in X_i \text{ and } x_j \in X_j\}.$$

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- This attack works for practical parameter sizes.

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- What can be said about logarithmic signatures of finite elementary abelian 2-groups?
- Related to perfect codes: Cohen, Litsyn, Vardy, Zémor 1996.

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- We've reviewed some of the cryptographic primitives based on groups (finite or infinite).
- We've sketched some problems with the security of MST_3 .
- Logarithmic signatures have applications to tiling problems; to perfect Gray codes and to other combinatorial problems in computer science.
- Can MST_3 , or any other group-theoretic cryptosystem, be made both secure and practical?

Some Links

This talk will appear soon on my home page:

<http://www.ma.rhul.ac.uk/sblackburn>

S.R. Blackburn, C. Cid, C. Mullan, 'Group theory in cryptography' (*Proc. Groups St Andrews at Bath 2009*, to appear) is available at:

<http://arxiv.org/abs/0906.5545>

The paper 'Cryptanalysis of the MST_3 public key cryptosystem' (*J. Math Cryptology*, 2010) is also available at:

<http://eprint.iacr.org/2009/248>